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# AN EFFECTIVE METHOD OF INVESTIGATING THE OSCILLATIONS OF SUBSTANTIALLY INHOMOGENEOUS DISTRIBUTED SYSTEMS†

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The problem of determining the natural frequencies and modes for one-dimensional substantially inhomogeneous distributed systems with boundary conditions of the first kind is investigated. Using the Rayleigh–Ritz method (in particular, the Rayleigh principle) an original determination of the small parameter is given and the corresponding perturbed boundary-value problem is formulated. A constructive method of obtaining bilateral estimates of the first eigenvalue (frequency) and for calculating the frequencies and modes with as high a specified degree of accuracy as desired is developed. The high efficiency of the proposed numerical–analytical method is illustrated by calculating specific examples which model the oscillations of an inhomogeneous string and by comparing the results obtained with available results. The analytic procedure and the computational algorithm are modified for the case of boundary conditions of the second kind. © 1997 Elsevier Science Ltd. All rights reserved.

## 1. FORMULATION OF THE PROBLEM

One-dimensional oscillatory systems with substantially inhomogeneous distributed parameters are considered (see [1–4], etc.). Mechanical models of such systems are a string, a continuous spring, an elastic beam or shaft, and certain hydrodynamic and quantum-mechanical systems. In applied problems the inertial and elastic properties may usually vary considerably along the length (with respect to the spatial coordinate), which is assumed to be finite. These characteristics include, for example, the linear density, distributed stiffness for stretching or bending, the density of a stratified liquid and a perturbing potential. In addition, an oscillatory distributed system may be in an elastic medium (a Winkler foundation [2, 3]), whose stiffness coefficient may also vary.

We will consider mainly the case of clamped left and right ends (boundary conditions of the first kind [3, 4]). The main results can then be formulated, without detailed derivation, for cases when one or both ends are free (boundary conditions of the second kind [3, 4]). The initial fairly smooth distributions of the displacements and velocities are assumed to be specified and matched with the boundary conditions, which may be time-varying. In general, there is a distributed external action (a force or moments of forces) on the system.

The linear initial-boundary-value problem considered admits of a separation of variables and can be reduced to determining the natural frequencies and modes of oscillations ([1–4], etc.). Using the methods of functional analysis and mathematical physics it is required to construct a solution of the self-conjugate boundary-value problem for eigenvalues and eigenfunctions (the Sturm–Liouville problem) [3–8].

Taking the above assumptions into account, the problem in question can be written as follows in dimensionless form

$$[p(x)\,u']' + [\lambda r(x) - q(x)]\,u = 0, \quad 0 < x < 1$$
$$u(0) = u(1) = 0, \quad u = u(x, \lambda) \tag{1.1}$$

Here $u$ is a function which defines the mode of the oscillations while $\lambda$ is the constant of the separation of variables ($\lambda = \omega^2$, where $\omega$ is the frequency). The specified fairly smooth functions $p$, $q$ and $r$ define the stiffness and inertial properties, respectively. They satisfy the constraints $0 < p_1 \leqslant p \leqslant p_2 < \infty$, $0 < r_1 \leqslant r \leqslant r_2 < \infty$, $q \geqslant 0$ for all $0 \leqslant x \leqslant 1$. The conditions imposed have a definite physical meaning. Note that the length of the system is normalized and is equal to unity. Moreover, to fix our ideas, for the present we will investigate the case of clamped ends.

It should also be noted that the requirement that the function $p(x)$ should be continuously differentiable according to (1.1) is redundant, since it is the characteristic $p(x)\,u' = z$ that has physical

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meaning. Equation (1.1) can be reduced to the form of the system $u' = z/p, z' = -(\lambda r - q)u$ with zero boundary conditions for $u$. Further, using the linear transformation of the function $v = up^{-1/2}(x)$ or a change of the argument $x \to y$ of the form $y = \int p^{-1}(x)\,dx$, Eq. (1.1) can be reduced to a form in which $p(x) \equiv 1$ or $p(y) \equiv 1$ [6].

For problem (1.1) it is required to obtain the values of the parameter $\lambda = \lambda_n$, $n = 1, 2, \ldots$ (the eigenvalues) for which non-trivial solutions $u_n(x) = u(x, \lambda_n)$ (eigenfunctions) exist, defining the natural frequencies and modes of the oscillations, respectively ([1–8], etc.). For applications the problem of finding the lowest frequencies and modes ($n \leqslant 10$) which, as a rule, define the performance of the mechanical system is of considerable importance; the calculation of $\lambda_1$, $u_1(x)$ is of particular interest. Our research was mostly devoted to this problem. The eigenvalues and functions with large numbers $n \gg 1$ can be calculated extremely effectively using asymptotic formulae [5].

When the functions $p$, $r$ and $q$, which may vary considerably in the range $x \in [0, 1]$ considered (for example, by many orders of magnitude), behave in a complex way, available methods may turn out to be insufficiently effective when determining the natural frequencies and modes and, primarily, the lower ones. To calculate these approximately, a number of methods have been developed including extremely sophisticated ones [1–5, 8–11]. As computational practice shows, the simplest and most natural Rayleigh–Ritz variational method often gives acceptable results for obtaining upper limits of lower frequencies. The method has been justified by Krylov [9] (convergence to the exact values was established and an estimate of its rate was given).

The construction of lower estimates is extremely important for applications. Results have been obtained which require fairly lengthy calculations, based on an approximate solution of Fredholm's equation of the second kind [10, 11], equivalent to the initial problem.

Thus, we will consider the problem of finding the first eigenvalue $\lambda_1$ of problem (1.1) as being of special interest from the theoretical and applied points of view. We will use the few-coordinate approximation of the eigenfunctions using the Rayleigh–Ritz method [1–3, 8–11]. We will formulate the variational problem equivalent to the Sturm–Liouville problem (1.1). It involves minimizing the quadratic functional $J[u]$ with the additional normalization condition $\Phi[u] = 1$ in the class of functions $u$ which satisfy the boundary conditions (1.1)

$$J[u] = \int_0^1 [p(x)u'^2 + q(x)u^2]\,dx \to \min \tag{1.2}$$

$$\Phi[u] = \|u\|_r^2 = (u,u)_r = \int_0^1 u^2 r(x)\,dx = 1, \quad u(0) = u(1) = 0$$

The solution $u_1(x)$ of variational problem (1.2) is the first eigenfunction of the Sturm–Liouville problem (1.1), while the minimum of the functional $J[u_1] = \lambda_1$ is the first eigenvalue.

To obtain the second eigenfunction $u_2(x)$ and eigenvalue $\lambda_2$ we have to solve problem (1.2) in a narrower class of functions $u$ which satisfy the additional condition of orthogonality $\Phi_1[u] = (u, u_1)_r = 0$, where the parentheses, as in (1.2), denote the scalar product of the functions $u$ and $u_1$ with weight $r(x)$. After the function $u_2(x)$ is constructed and the eigenvalue $\lambda_2 = J[u_2]$ has been obtained, we similarly formulate the problem for determining $u_3(x)$, $\lambda_3$, and also the subsequent functions $u_n(x)$ and eigenvalues $\lambda_n$

$$J[u] \to \min, \quad \Phi[u] = 1, \quad \Phi_k[u] = (u, u_k)_r = 0, \quad k = 1, 2, \ldots, n-1 \tag{1.3}$$

$$J[u_n] = \lambda_n, \quad u_n(x) = u(x, \lambda_n)$$

The procedure for constructing subsequent solutions of the Sturm–Liouville problem (1.1) by reduction to variational problems with equality constraints of the type (1.3) can be continued without limit.

We will further consider problem (1.2) as the least complicated one and of practical interest. In applications, the thoroughly developed Rayleigh–Ritz method [1–11] is often used for its approximate solution. The Rayleigh principle is a special result; this enables one to obtain fairly simply, using quadratures, an upper limit $\lambda_1^*$ for the first eigenvalue $\lambda_1$

$$0 < \lambda_1 \leqslant \lambda_1^* = \frac{J[\psi_1]}{\Phi[\psi_1]}, \quad \psi_1(0) = \psi_1(1) = 0 \tag{1.4}$$

Here $\psi_1(x)$ is a certain ("test") function, not necessarily normalized by (1.2), having a continuous first derivative for $x \in [0, 1]$. It is chosen from additional considerations regarding the first form of the oscillations [1, 3, 9]. In particular, the function $\psi_1 \neq 0$ when $0 < x < 1$, i.e. it has no intermediate zeros; when $q \equiv 0$ it is convex upwards or downwards, etc.

The equality $\lambda_1 = \lambda_1^*$ in (1.4) holds if and only if $\psi_1(x) = ku_1(x)$, $k = \text{const}$. Estimate (1.4) does not enable us to judge how close $\lambda_1^*$ and $\lambda_1$ are. To do this we need to obtain the lower estimate $\lambda_1^* \geqslant \lambda_1$, which is a rather difficult problem. As it turned out, available approaches [10, 11] to constructing such estimates use the theory of Fredholm integral equations of the second kind. They require complicated multiple quadratures, which lead to considerable computational difficulties. In view of what was said above it is important to study the following problems.

1. To establish how well the limit $\lambda_1^*$ of the type (1.4), obtained by the Rayleigh–Ritz method, using a small set of coordinate functions $\psi_i(x)$ (in particular, according to the Rayleigh principle using one function $\psi_i(x)$), approximates to the exact value $\lambda_1$.
2. To obtain a lower limit $\lambda_{1*}$, which, together with $\lambda_1^*$, gives a satisfactory estimate of $\lambda_1$.
3. To refine the limits $\lambda^*$, $\lambda_1$ obtained.
4. To develop a procedure for successively refining the estimates, which gives, in the limit, the required accurate value of $\lambda_1$.
5. To suggest approaches for the approximate solution of other boundary-value problems.

## 2. THE LOWER LIMIT OF THE FIRST EIGENVALUE

Suppose $\psi_1(x)$ is an admissible function, i.e. continuously differentiable and satisfying boundary conditions (1.4); it is not necessary to satisfy the normalization condition with weight $r(x)$ (1.2). Suppose $\lambda_1^*$ is the upper limit of the eigenvalue $\lambda_1$, obtained using Rayleigh's principle, see (1.2). Together with the Sturm–Liouville problem (1.1) ($n = 1$) we will consider the following Cauchy problem

$$[p(x)v']' + [\lambda_1^* r(x) - q(x)]v = 0, \quad v(0) = 0, \quad v'(0) = 1 \tag{2.1}$$

Its solution $v = v_1(x, \lambda_1^*)$ is assumed to be known in any of the following forms: analytic, numerical or in the form of an algorithm (a procedure). It can then be established that the function $v_1$ possesses the following properties

$$v_1(0, \lambda_1^*) = v_1(\xi, \lambda_1^*) = 0, \quad 0 < \xi \leqslant 1 \tag{2.2}$$

Here $\xi$ is the first point of intersection of the $x$ axis (the abscissa). The right-hand side of inequality (2.2) for $\xi$ is a consequence of the second Sturm oscillation theorem [6], where the equality only occurs when $\lambda_1^* = \lambda_1$. Hence, in the general case when $\psi_1 \neq ku_1$ we obtain $\lambda_1 < \lambda_1^*$, $0 < \xi < 1$. The smallness of the quantity $\varepsilon = 1 - \xi > 0$ can serve as a measure of the closeness of the upper estimate $\lambda_1^* - \lambda_1 > 0$, which is ensured on the basis of the results obtained in [9].

We will correct (reduce) the quantity $\lambda = \lambda_1^*$ in (2.1) by multiplying it by the number $\varkappa$, $0 < \varkappa < 1$. It is natural to relate the quantity $\varkappa$ to $\xi < 1$, for example, $\varkappa = \varkappa(\xi) = \xi^\gamma$, $0 < \gamma < \infty$, see below. We will now consider the Cauchy problem of the type (2.1) with a reduced value of $\lambda = \lambda_1^* \varkappa$

$$[p(x)w']' + [\lambda_1^* \varkappa r(x) - q(x)]w = 0, \quad w(0) = w'(0) = 1 \tag{2.3}$$

Suppose the solution $w = w_1(x, \lambda_1^* \varkappa)$, $x \in [0, 1]$ of the Cauchy problem (2.3) is constructed numerically or analytically. We have the following assertion.

*Theorem* 1. If $w = w_1(x, \lambda_1^* \varkappa) \geqslant 0$ when $0 < x \leqslant 1$, the quantity $\lambda_1^* \varkappa = \lambda_{1*}$ gives the lower limit of the first eigenvalue

$$\lambda_{1*} \leqslant \lambda_1 \leqslant \lambda_1^*, \quad \lambda_{1*} = \lambda_1^* \varkappa \tag{2.4}$$

The upper limit $\lambda_1^*$ was constructed previously using the Rayleigh principle (1.4) (or the Rayleigh–Ritz principle, see Section 5).

The proof of the assertion follows directly from the properties of the first eigenfunction $u_1(x)$. In fact, for $u_1(x)$ when $\lambda = \lambda_1$ we have the identity (1.1), where, to be specific, we can assume $u_1'(0) > 0$; then $u_1'(1) < 0$, $u_1(x) = 0$, $0 < x < 1$. The quantities $u_1(x)$, $\lambda_1$ are not assumed to be known. Similarly, for

the known function $w_1$—the solution of the Cauchy problem (2.3)—a similar identity holds. Multiplying the identity for $u_1$ by $w_1$ and the identity for $w_1$ by $u_1$, subtracting and integrating by parts, taking the boundary conditions into account, we obtain the equality

$$(\lambda_1 - \lambda_1^* \varkappa) \int_0^1 u_1 w_1 r\, dx = -p(1)\, u_1'(1)\, w_1(1, \lambda_1^* \varkappa) \tag{2.5}$$

It follows from the condition of the theorem that the right-hand side of (2.5) is non-negative. Since the integral is positive, the lower limit $\lambda_{1*}$ (2.4) follows from (2.5).

We will discuss the constructive aspect of the theorem. It is apparently small, as in the "shooting" method, unless we assume an effective rule for choosing $\varkappa$, see Section 3. The convenience of using it is due to the fact that the abscissa $\xi$ is determined by direct analytic or numerical integration of the Cauchy problem (2.1).

## 3. THE USE OF THE PERTURBATION METHOD TO REFINE THE SOLUTION OF THE STURM–LIOUVILLE PROBLEM

It follows from (2.2) and the estimate (2.4) that the criterion for the accuracy of the few coordinate (in particular, the single-coordinate $\psi_1(x)$) approximation of the first eigenfunction $u_1(x)$ is the satisfaction of the strong inequality

$$1 - \xi = \varepsilon \ll 1, \quad \varepsilon > 0 \tag{3.1}$$

The smallness of the numerical parameter $\varepsilon$ (3.1) is achieved by appropriate choice of the test function $\psi_1(x)$ or on the basis of the approximation in [9]. This property enables an effective procedure to be developed for refining the solution of problem (1.1) and the limits $\lambda_1^*$, $\lambda_{1*}$.

We will introduce a new independent variable $y = \xi x$ and convert relations (1.1) to the form of a perturbed Sturm–Liouville problem with a small parameter

$$\left[p\left(\frac{y}{\xi(\varepsilon)}\right)u'\right]' + \left[\mu r\left(\frac{y}{\xi(\varepsilon)}\right) - \frac{1}{\xi^2(\varepsilon)}\, q\left(\frac{y}{\xi(\varepsilon)}\right)\right]u = 0, \quad \mu = \frac{\lambda}{\xi^2} \tag{3.2}$$

$$u = u(y, \varepsilon), \quad 0 \leqslant y \leqslant \xi, \quad u(0, \varepsilon) = u(\xi, \varepsilon) = 0, \quad \frac{1}{\xi(\varepsilon)} = 1 + \varepsilon + \varepsilon^2 + \ldots$$

The primes denote derivatives with respect to $y$. This method is analogous to the one proposed by Poincaré [12, 13] for constructing perturbed periodic motions of non-linear oscillatory systems.

We will take the following as the "generating problem"

$$[p(y)\, u']' + [\mu r(y) - q(y)]\, u = 0, \quad u(0) = u(\xi) = 0 \tag{3.3}$$

According to (2.1) and (2.2) the first eigenvalue $\mu_1 = \lambda_1^*$ and the function $u_1 = v_1(y, \lambda_1^*)$ are known. Using these when $\varepsilon > 0$ is sufficiently small, we can develop an effective procedure of the perturbation method which enables the required solution $\mu_1(\varepsilon)$, $u_1(y, \varepsilon)$ of problem (3.2) to be constructed with an arbitrary specified degree of accuracy with respect to $\varepsilon$.

We will initially use a standard scheme or the expansion of the required quantities in powers of $\varepsilon$. Assuming the functions $p(x)$, $r(x)$, $q(x)$ to be continuous, we reduce the perturbed Sturm–Liouville problem (3.2) to the form

$$[p(y)\, u']' + [\mu r(y) - q(y)]\, u = \varepsilon H(\mu, y, \varepsilon, [u]), \quad u(0, \mu, \varepsilon) = u(\xi, \mu, \varepsilon) = 0 \tag{3.4}$$

$$H \equiv -[p'yu']' - \mu y r'(y)\, u + 2q(y)\, u + yq'(y)\, u + \varepsilon \ldots, \quad \mu = \lambda \xi^{-2}$$

Here $\mu = \mu_1(\varepsilon)$ is the required eigenvalue and $u = u_1(y, \mu, \varepsilon)$ is the corresponding function. We will seek the solution of problem (3.4) in the form of expansions

$$\mu = \mu_1 + \varepsilon \nu_1^{(1)} + \varepsilon^2 \ldots, \quad u = u_1 = v_1(y, \lambda_1^*) + \varepsilon v_1^{(1)}(y) + \varepsilon^2 \ldots \tag{3.5}$$

Substituting (3.5) into (3.4) and equating coefficients of like powers of $\varepsilon$, we obtain a sequence of boundary-value problems of the same type

$$[p(y)v_1^{(i)}]' + [\lambda_1^* r(y) - q(y)]v_1^{(i)} = -\nu_1^{(i)} r(y) v_1 + H_i(y) \tag{3.6}$$
$$v_1^{(i)}(0) = v_1^{(i)}(\xi) = 0, \quad i \geqslant 1$$

Here at the $i$th step the unknowns are $\nu_1^{(i)}$, $v_1^{(i)}$, while the functions $H_i(y)$ are determined from the solutions at previous steps. The solution of problem (3.6) is constructed by standard methods of the theory of linear differential equations [6]. This can be carried out in the form of quadratures, since the general solution of the corresponding homogeneous equation is found using Liouville's formula, see Section 4, and the known function $v_1(y, \lambda_1^*)$.

For boundary-value problem (3.6) to be solvable it is necessary (the Fredholm alternative) for the right-hand side to be orthogonal to $v_1$. From this relation, at each $i$th step, the required coefficients $\nu_1^{(i)}$ are determined

$$\nu_1^{(i)} = \frac{1}{\|v_1\|^2} \int_0^\xi H_i(y) v_1(y, \lambda_1^*)\, dy, \quad \|v_1\|^2 = \int_0^\xi v_1^2(y, \lambda_1^*)\, r(y)\, dy \tag{3.7}$$

Note that the function $H_1$, by (3.4), is found using the known $\lambda_1^*$ and $v_1(y, \lambda_1^*)$. From (3.7) we then obtain an expression for $\nu_1^{(1)}$

$$\nu_1^{(1)} = \frac{1}{\|v_1\|^2} \int_0^\xi \{p'(y)\, y v_1'^2 + [-\lambda_1^* y r'(y) + 2q(y) + y q'(y)] v_1^2(y, \lambda_1^*)\}\, dy \tag{3.8}$$

As a result we obtain the required first number $\mu_1(\varepsilon)$ with error $O(\varepsilon^2)$, see (3.5). From the definition of $\mu$ (3.4) we obtain $\lambda_1(\varepsilon)$ with the same error $O(\varepsilon^2)$

$$\lambda_1 = \lambda_1(\varepsilon) = \lambda_1^* \xi^2 \left(1 + \varepsilon \frac{\nu_1^{(1)}}{\lambda_1^*}\right) + O(\varepsilon^2) = \lambda_1^* \left[1 + \varepsilon \left(\frac{\nu_1^{(1)}}{\lambda_1^*} - 2\right)\right] + O(\varepsilon^2) \tag{3.9}$$

Higher approximations of the required quantities $\lambda_1(\varepsilon)$, $u_1(x, \varepsilon)$ are constructed and the convergence is proved in Section 4.

We will once again discuss the results of Theorem 1 of Section 2 and consider the lower limit $\lambda_{*1}$ (2.4) taking into account expression (3.9) for $\lambda_1$. If $\varepsilon$ is fairly small, then when $\nu_1^{(1)} > 0$, see (3.8), we obtain $\lambda_{*1} = \lambda_1^* \xi^2 (1 + \varepsilon\eta\nu_1^{(1)})$ for $\lambda_{*1}$, where $\eta < 1$, and in particular $\eta = 0$, i.e. $\lambda_{*1} = \lambda_1^* \xi^2$. The rough condition which leads to inequality $\nu_1^{(1)} > 0$ is the fact that the expressions in braces in (3.8), in particular $p' \geqslant 0$, $r' \leqslant 0$, $q' \geqslant 0$ are non-negative (see the calculation of the examples in Section 5). Note that when $\nu_1^{(1)} > 0$ for $\varkappa = \xi^2 (\gamma = 2)$ the condition $w_1 \geqslant 0$ of Theorem 1 must be satisfied.

Consider the case when $\nu_1^{(1)} < 0$. Then the lower limit is the quantity $\lambda_{1*} = \lambda_{1*} \xi^2 (1 + \varepsilon\eta\nu_1^{(1)})$, $\eta > 1$. The quantity $\lambda_1^{*(2)} = \lambda_1^* \xi^{(2)}$ will be an improved upper limit, which can be used in the same way as $\lambda_1^*$ (see Sections 2 and 3). The process of constructing the lower limit can be continued (see Example 2 in Section 5).

If there is a critical case $\nu_1^{(1)} = 0$, the lower limit will be $\lambda_{1*} = \lambda_1^* \xi^{(2)} \eta$, $\eta < 1$ ($\eta = 1 - O(\varepsilon)$). A more accurate estimate requires calculations of the following coefficients $\nu_1^{(i)}$, $i \geqslant 2$. Hence, when $\varepsilon \ll 1$ the lower limit will certainly be the quantity $\lambda_{1*} = \eta\lambda_1^*$, $1 - \eta = \delta$, $1 \gg \delta \gg \varepsilon$ (for example, $\delta = \varepsilon^\chi$, $0 < \chi < 1$, in particular $\delta = \varepsilon^{1/2}$) irrespective of the sign of $\nu_1^{(1)}$.

An approach similar to that used above can be employed and extended to boundary conditions of the second kind (see Section 6) and the third kind, and also to obtain the second eigenvalues $\lambda_2$ and function $u_2$ of problem (1.1) and the following ones in accordance with (1.3).

An investigation of the problem of constructing improved estimates and a refined solution of boundary-value problem (3.2) is of particular interest from the theoretical and applied points of view. In what follows, we describe an approach which is based on the use of the method of successive approximations, developed to investigate problems of the theory of non-linear oscillations [12, 13, etc.].

## 4. THE CONSTRUCTION OF AN EXACT SOLUTION AND THE JUSTIFICATION OF THE PERTURBATION METHOD

The non-standard procedure for obtaining the small parameter and the use of the perturbation method for the approximate solution of boundary-value problem (3.2) or (3.4), described above, can be extended by taking into account higher powers of $\varepsilon^j, j \leq 1$. However, the approach described in Section 3 cannot be adapted to computer calculations of higher approximations. Hence, it is best to convert the initial perturbed Sturm–Liouville problem (3.2) to the following form (to simplify the calculations we here assume $p(x) \equiv 1$, which does not reduce the generality, see Section 1)

$$u'' + [\mu r(y) - q(y)]\, u + h(\mu, y, \varepsilon)\, u = 0$$
$$u = u(y, \mu, \varepsilon), \quad u = u(0, \mu, \varepsilon) = u(\xi, \mu, \varepsilon) = 0, \quad y \in [0, \xi] \tag{4.1}$$
$$\mu = \lambda d^2, \quad d = (1-\varepsilon)^{-1}, \quad h \equiv \mu[r(dy) - r(y)] + q(y) - d^2 q(dy)$$

Assuming the function $r(x), q(x), x \in [0, 1]$ to be continuous, we obtain that $h = O(\varepsilon)$ for $y \in [0, \xi]$, $\mu \sim \lambda_1^* \sim 1$. Since system (4.1) is linear in $u$ and $\mu$ (bilinear in the combination $u$, $\mu$), henceforth it will be sufficient to confine ourselves to the Lipschitz condition with respect to $x$ for $r$ and $q$. Assuming $\varepsilon = 0$ in (4.1), we obtain the unperturbed problem (3.3), the solution $\mu_0 = \lambda_1^*$, $u_0(y, \mu_0) = v_1(y, \lambda_1^*)$ of which—the first eigenvalue and eigenfunction—is assumed to be known (see (2.1), (2.2) and (3.3)).

A direct solution of boundary-value problem (4.1) by methods of perturbation theory turns out to be rather difficult and time consuming because of the need to satisfy the boundary condition at $y = \xi$. Hence, we will use its corollary

$$\mu \int_0^\xi u^2 r(y)\, dy = \int_0^\xi \{u'^2 + [q(y) - h(\mu, y, \varepsilon)]\, u^2\}\, dy \tag{4.2}$$

and dispense with the indicated boundary condition (4.1). We replace it by the arbitrary non-zero condition for the derivative $u'$ when $y = 0$, in particular, we put $u'(0, \mu. \varepsilon) = 1$. We assume this change, since $u'(0, \mu. \varepsilon) \neq 0$, and, moreover, the functions $u$ are found to within a non-zero factor.

Thus, instead of the initial perturbed boundary-value problem (4.1) we consider the Cauchy problem

$$u'' + [\mu r(y) - q(y)]\, u + h(\mu, y, \varepsilon)\, u = 0 \tag{4.3}$$
$$u(0, \mu, \varepsilon) = 0, \quad u'(0, \mu, \varepsilon) = 1$$

which, together with Eq. (4.2), enables us to obtain the required quantities $\mu = \mu_1(\varepsilon)$, $u = u_1(y, \varepsilon)$—the first eigenvalue and eigenfunction, respectively, more simply. We will apply the standard procedure of the method of successive approximations to the equivalent problem (4.2), (4.3). For this purpose we make the following change of variables

$$\mu = \mu_0 + \nu, \quad \mu_0 = \lambda_1^*; \quad u = u_0 + z, \quad u_0(y, \mu_0) = v_1(y, \lambda_1^*) \tag{4.4}$$

Relation (4.2), after substituting expressions (4.4), can be reduced to a quasi-linear functional equation of the form

$$\nu = \nu_1^* + N(\nu, [z], [z'], \varepsilon), \quad \nu_1^* = \nu_1^*(\varepsilon) = -\|u_0\|^{-2} I[h_0 u_0^2]$$
$$\|u_0\|^2 = I[u_0^2 r], \quad I[\varphi] \equiv \int_0^\xi \varphi(y)\, dy, \quad N = I[\Psi(\nu, z, z', y, \varepsilon)] \tag{4.5}$$
$$\Psi \equiv z'^2 + (q - \mu_0 r)\, z^2 - (h_0 + \nu r)(2u_0 z + z^2) - \nu h_\nu (u_0 + z)$$
$$h_0 \equiv h(\mu_0, y, \varepsilon), \quad h_\nu \equiv r(dy) - r(y), \quad h_\nu \sim \varepsilon$$

Since the quantity $\mu_0 = \lambda_1^*$—the first eigenvalue of the unperturbed problem—is known, we will not indicate the dependence of $u_0$ on $\mu_0$ to simplify the notation. Further, the square brackets in (4.5) denote

that the operator of integration with respect to $y$, similar to the operator $I$, is applied to the corresponding functions.

We can similarly transform the Cauchy problem (4.3) for the perturbation of $z$ (4.4)

$$z'' + [\mu_0 r(y) - q(y)]\, z = -(\nu_1^* r + h_0)\, u_0 - F(\nu, z, [z], [z'], y, \varepsilon) \tag{4.6}$$
$$F \equiv (\nu r + h)\, z - \nu(h_\nu + N u_0 r); \quad z(0, \nu, \varepsilon) = 0, \quad z'(0, \nu, \varepsilon) = 0$$

When $\varepsilon = 0$, problem (4.5), (4.6) admits of a trivial solution. The functions and operators $N$ and $F$ are of quadratic order with respect to the small quantities $\varepsilon$, $\nu$, $\nu_1$, $z$, $z'$. We will construct the required solution $\nu = \nu(\varepsilon)$, $z = z(y, \varepsilon)$ by successive approximations using the recurrence scheme [13]

$$\nu_{j+1} = \nu_1^* + N(\nu_j, [z_j], [z_j'], \varepsilon), \quad j \geqslant 1; \quad \nu_1 = \nu_1^*(\varepsilon) \sim \varepsilon$$
$$z_{j+1} = z_1^*(y, \varepsilon) + L(\nu_j, [z_j], [z_j'], y, \varepsilon), \quad z_{j+1}' = \frac{dz_{j+1}}{dy}$$
$$z_1^*(y, \varepsilon) \equiv -\int_0^y W(y, s)[\nu_1^* r(s) + h(\mu_0, s, \varepsilon)]\, u_0(s)\, ds; \quad z_1^*, z_1^{*\prime} \sim \varepsilon \tag{4.7}$$
$$L_j \equiv -\int_0^y W(y, s)\, F(\nu_j, z_j(s, \varepsilon), [z_j], [z_j'], s, \varepsilon)\, ds; \quad L_j, L_j' \sim \varepsilon^2$$

Here $W(y, s)$ is the unit impulse response function of the unperturbed equation (4.6). It is known since it is possible to construct a general solution $z_0$ of the homogeneous equation using Liouville's formula [6, 13]

$$z_0 = c_1 u_0(y) + c_2 w_0(y), \quad w_0(y) \equiv u_0(y) \int_0^y \frac{ds}{u_0^2(s)} \tag{4.8}$$
$$W(y, s) = -u_0(y)\, w_0(s) + w_0(y)\, u_0(s), \quad W(s, s) = 0, \quad W_y'(s, s) = 1$$

The function $W(y, s)$ (4.8) is uniformly bounded for all $y, s, \xi \geqslant y \geqslant s \geqslant 0$ [6].

*Theorem* 2. Successive approximations of $\nu_j, z_j, z_j'$ (4.7) converge uniformly as $j \to \infty$ for sufficiently small values of $\varepsilon$, $\varepsilon \in [0, \varepsilon_0]$ to a unique exact solution $\nu^*(\varepsilon)$, $z^*(y, \varepsilon)$, $z^{*\prime}(y, \varepsilon)$ of problem (4.5), (4.6) which vanishes at $\varepsilon = 0$.

The proof is divided into several stages. First, we establish by induction that the successive approximations (4.7) are uniformly bounded irrespective of the number $j = 1, 2, \ldots, k, k+1, \ldots$ when $\varepsilon > 0$ is fairly small. Hence, we have the inequalities

$$|\nu_j| \leqslant \varepsilon c_\nu, \quad |z_j| \leqslant \varepsilon c_z, \quad |z_j'| \leqslant \varepsilon c_{z'}, \quad j \geqslant 1 \tag{4.9}$$

in which the constants $c_\nu, c_z, c_{z'}$ are found constructively using the estimates of the functions $W, r, q, h, h_\nu$.

Inequalities (4.9) then enable us to establish, secondly, that the operators (4.7) "reduce the distance". Using Banach's theorem on the contraction operator, one can establish the existence of a fixed point (see [7, 8]). Thus, the existence of the limits $\nu^*(\varepsilon)$, $z^*(y, \varepsilon)$, $z^{*\prime}(y, \varepsilon)$ for sufficiently small values of $\varepsilon > 0$ has been proved. These limits are bounded by (4.9) and vanish when $\varepsilon = 0$.

The value of the parameter $\varepsilon_0$, which defines the region of convergence with respect to $\varepsilon$, can be estimated constructively as follows [7, 8, 13]. For the differences

$$\Delta\nu_{j+1} = |\nu_{j+1} - \nu_j|, \quad \Delta z_{j+1} = \max_{0 \leqslant y \leqslant \xi} |z_{j+1} - z_j|, \quad \Delta z_{j+1}' = \max_{0 \leqslant y \leqslant \xi} |z_{j+1}' - z_j'|$$

from (4.7) using the limits (4.9) we obtain inequalities of the form

$$\Delta g_{j+1} \leqslant \varepsilon(\alpha_g \Delta\nu_j + \beta_g \Delta z_j + \gamma_g \Delta z_j'), \quad g = \nu, z, z' \tag{4.10}$$

The constants $(\alpha, \beta, \gamma)_g$ in (4.10) are effectively defined in terms of the right-hand sides of relations

(4.7), similar to $c_g$ in (4.9). Adding inequalities (4.10) we obtain the estimate

$$\Delta_{j+1} \leqslant \varepsilon C \Delta_j; \quad \Delta_j \equiv \Delta\nu_j + \Delta z_j + \Delta z'_j, \quad C = \max(A, B, \Gamma) \tag{4.11}$$

$$A = \alpha_\nu + \alpha_z + \alpha_{z'}, \quad B = \beta_\nu + \beta_z + \beta_{z'}, \quad \Gamma = \gamma_\nu + \gamma_z + \gamma_{z'}$$

The main condition of the theorem on the contraction operator follows from (4.11), namely

$$\varepsilon_0 C = \theta < 1, \quad \varepsilon_0 = \theta C^{-1}, \quad \varepsilon \in [0, \varepsilon_0] \tag{4.12}$$

The convergence $\nu_j \to \nu^*$, $z_j \to z^*_j$, $z'_j \to z^{*\prime}$ for condition (4.12) follows from the property of the absolute and uniform convergence as $j \to \infty$ of the partial sums of the form $\nu_{j+1} = \nu_1 + (\nu_2 - \nu_1) + \ldots + (\nu_{j+1} - \nu_j)$ and similar sums for $z_{j+1}, z'_{j+1}$. Since we have the estimates (4.11) for the terms, the corresponding series converge as $j \to \infty$, as the sums of terms of a geometric progression with denominator not greater than $\theta$, $\theta < 1$.

Thirdly, we can prove that the limiting functions $\nu^*(\varepsilon)$, $z^*(y, \varepsilon)$, $z^{*\prime}(y, \varepsilon)$ satisfy Eqs (4.5) and (4.6). The Cauchy problem (4.6) must be represented using the response function $W(y, s)$ in the form of an integral equation with operator $L$ (see (4.7)). Further we substitute these functions into $N$, $F$ and $L$; we obtain certain values of $\nu, z, z'$. Using (4.7) we calculate $\max \Delta g_{j+1}$. They satisfy inequalities of the type (4.10) in which the expressions on the right-hand sides contain the differences $\Delta^* g_j$ instead of $\Delta g_j$. Since the latter, as was established above, tend to zero as $j \to \infty$, this indicates that

$$\nu = \nu^*(\varepsilon), \quad z = z^*(y, \varepsilon), \quad z' = z^{*\prime}(y, \varepsilon), \quad y \in [0, \xi], \quad \varepsilon \in [0, \varepsilon_0] \tag{4.13}$$

is the solution of problem (4.5), (4.6). According to formulae (4.4) the functions

$$\mu = \mu_0 + \nu^*(\varepsilon), \quad u = u_0(y) + z^*(y, \varepsilon), \quad u' = du/dy \tag{4.14}$$

$$\lambda = d^{-2}[\mu_0 + \nu^*(\varepsilon)], \quad u = u_0(d^{-1}x) + z^*(d^{-1}x, \varepsilon), \quad u'_x = d^{-1}du/dy$$

are solutions of the Sturm–Liouville problems (4.1) and (1.1), respectively.

Finally, and fourthly, we will prove that the solution (4.13), and, hence, also the solution (4.14), is unique for sufficiently small $\varepsilon > 0$, see (4.12). Suppose we assume the opposite, i.e. $\nu_*, z_*, z_*'$ is another solution of problem (4.5), (4.6), which satisfies (4.9). Then, for the differences $\Delta\nu = \nu^* - \nu_*$, $\Delta z = |z^* - z_*|_{\max}$, $\Delta z' = |z^{*\prime} - z_*|_{\max}$ we obtain inequalities of the type (4.12), where, on the right-hand sides we have the same quantities $\Delta\nu, \Delta z, \Delta z'$. If the strict inequality (4.12) is satisfied, these inequalities can only hold provided $\Delta\nu = 0$, $\Delta z = 0$, $\Delta z' = 0$, i.e. $\nu^* \equiv \nu_*$, $z^* \equiv z_*$, $z^{*\prime} \equiv z'_*$. Hence, the uniqueness of solution (4.13) and (4.14) is also established.

Note that the required solution $\nu(\varepsilon)$, $z(y, \varepsilon)$, $z'(y, \varepsilon)$ can be constructed using the procedure of expansions in powers of the small parameter $\varepsilon$. Here the analyticity of relations (4.2) and (4.3) with respect to $\mu, u, u'$ or of (4.5) and (4.6) with respect to $\nu, z, z'$ is used. The analyticity of $r$ and $q$ with respect to $y$ and $\varepsilon$ is not required. The proof of the existence of a solution, the justification for the convergence of the series, the estimate of the radius of convergence and the proof of the uniqueness are carried out by the method of Cauchy majorant functions [3, 14].

Using the Rayleigh–Ritz method, it is possible, without any serious difficulties, to extend the approach described in Section 4 to the problem of constructing subsequent eigenvalues $\lambda_n$ and eigenfunctions $u_n(x)$, $n \geqslant 2$.

Hence, the proposed non-standard method of introducing a small numerical parameter $\varepsilon$ (3.1) enables the values of the eigenvalues and eigenfunctions to be refined. Unlike the shooting method, the refinement is carried out with respect to powers of $\varepsilon^j$, and not by successive division of the interval. Using (3.3), (3.8) and (3.9) we can construct a numerical–analytic procedure which accelerates the convergence [15, 16].

The effectiveness of the method is illustrated below by calculations of model examples. In the general case, for the numerical integration of the Cauchy problem and to evaluate the integrals it may be necessary to use a PC AT. However, for simple calculations satisfactory results can be obtained using programmable microcalculators.

## 5. MODEL EXAMPLES

To illustrate the effectiveness of the approaches described in Sections 2–4 we will consider boundary-value problems which arise when investigating the oscillations of an inhomogeneous string clamped at the ends [3, 4]. Using the method of separation of variables, we obtain Sturm–Liouville problems of the form (1.1) for the coordinate functions in which $p(x) \equiv 1$, $q(x) \equiv 0$.

*Example* 1. We take the function of the linear density in the form $r(x) = 1 + \sin \pi x, x \in [0, 1]$. Using the test function $\psi_1 = \sin \pi x$ we obtain, by (1.4), the upper limit of the first eigenvalue: $\lambda_1 \leqslant \lambda_1^* = 5.33827$. Integration of Cauchy's problem (2.1) leads to the abscissa $\xi = 0.999417$, which is very close to unity. The value of the small parameter $\varepsilon$ is found from (3.1): $\varepsilon = 1 - \xi = 5.83 \times 10^{-4} \ll 1$ and turns out to be fairly small. The corrected value $\lambda_1^* \xi^2 = 5.33205$, as follows from the results of integration of the corresponding Cauchy problem (2.3) (with $\varkappa = \xi^2, \gamma = 2$), leads to a quite accurate lower limit $\lambda_{1*}$ of the eigenvalue $\lambda_1$ (the conditions of Theorem 1 are satisfied). In addition, this assertion follows from (3.8), since $v_1^{(1)} > 0$.

Further, from multicoordinate extremely lengthy calculations and estimates [10, 11] we have as the "exact" value $\lambda_1 = 0.54032\pi^2 \simeq 5.33274$. The refined value obtained from (3.9) is $\lambda_1^{(1)} = 5.33284$, which leads to a relative error of $|\lambda_1^{(1)} - \lambda_1| \lambda_1^{-1} \sim 10^{-5}$, i.e. $O(\varepsilon^2)$.

*Example* 2. For comparison with the above results we will consider the apparently close density function $r(x) = 1 + \sin(\pi x/2), x \in [0, 1]$. Note that the mean density (the overall mass) of the string in both cases is the same and equal to $1 + 2/\pi$. We will use the same test function $\psi_1 = \sin \pi x$ in the Rayleigh principle; we obtain the upper limit $\lambda_1^* = 5.87805$. Integration of the Cauchy problem (2.1) leads to the value $\xi = \xi^{(1)} = 0.99805$, which is also very close to $x = 1$; the parameter $\varepsilon = 1 - \xi^{(1)} \simeq 2 \times 10^{-3}$. The analysis of Cauchy problem (2.3) with $\lambda_1^{*(2)} = \lambda_1^* \xi^{(1)2}$ shows that the conditions of Theorem 1 are not satisfied: $w_1(1, \lambda_1^{(2)}) < 0$. Moreover, as follows from (3.9), $v_1^{(1)} < 0$, which confirms these conclusions. Hence $\lambda_1^{*(2)} = \lambda_1^* \xi^{(1)2}$ will be an improved upper limit of the number: $\lambda_1^* > \lambda_1$ ($\lambda_1^{*(2)} < \lambda_1^*$), which we will use to determine the exact abscissa $\xi = \xi^{(2)}$. The analysis of Cauchy problem (2.1) with $\lambda = \lambda_1^{*(2)}$ leads to a closer value of the root $\xi$ (2.2): $\xi = \xi^{(2)} = 0.99975$ and to a considerably lower value of the small parameter $\varepsilon = 2.5 \times 10^{-4}$. This procedure can be extended to the third step to obtain once again an improved upper limit $\lambda_1^{*(3)} = \lambda_1^{*(2)} \xi^{(2)2} = 5.85222$, i.e. $5.85215 < \lambda_1 < 5.85222$ and a quite accurate value $\xi = \xi^{(3)} = 0.99999$ (more accurately, $\xi^{(3)} = 0.999994$). After the fourth iteration we obtain $\lambda_1^{(4)} = \lambda_1^{*(3)} \xi^{(3)2} = 5.85215$; by integrating Cauchy problem (2.3) we can finally establish numerically that $w_1(x, \lambda_1^{(4)}) > 0, x \in [0, 1]$, i.e. $\lambda_1^{(4)} = \lambda_{1*}^{(4)}$ is the lower limit. As a result we have obtained a quite sharp bilateral estimate $\lambda_{1*}^{(4)} < \lambda_1 < \lambda_1^{*(3)}$.

In the example considered we will take the more accurate two-coordinate approximation of the Rayleigh–Ritz method: $\psi_1(x) = c_1 \sin \pi x + c_2 \sin 2\pi x$. This test function leads to the upper limit $\lambda_1^* = 5.85232$ and a value of $\xi = 0.99999$ (more accurately, $\xi = 0.999985$). The value $\lambda_1^* \xi^2 = 5.85214$ turns out to be the lower limit since the conditions of Theorem 1 are satisfied and, moreover, $v_1^{(1)} > 0$. Hence, we have immediately established a satisfactory bilateral estimate $\lambda_1^* \xi^2 < \lambda_1 < \lambda_1^*$, but it is somewhat less accurate than that obtained above after the fourth iteration. Computational experience shows that it may be more economic to choose a somewhat more accurate test function [9].

*Example* 3. We will investigate another instructive example for which we can construct an exact closed-form solution of the Sturm–Liouville problem. We will take the function $r(x) = (1 + x^2)^{-2}, x \in [0, 1]$. The complete solution of the problem has the form

$$\lambda_n = 16n^2 - 1, \quad u_n(x) = \sqrt{1 + x^2} \sin(4n \operatorname{arctg} x), \quad n = 1, 2, \ldots$$

$$\lambda_1 = 15, \quad u_1(x) = \sqrt{1 + x^2} \sin(4 \operatorname{arctg} x) = 2x(1 - x^2)(1 + x^2)^{-3/2}$$

For the simplest test function $\psi_1 = \sin \pi x$, using the results obtained in Section 2, we obtain the required lower limit $\lambda_1^* = 15.33728$, the abscissa $\xi = 0.98350$ and the small parameter $\varepsilon = 1.65 \times 10^{-2}$. According to Theorem 1, we obtain the lower limit of $\lambda_1$: $\lambda_{1*} = \lambda_1^* \xi^2 = 14.83533$. Note, incidentally, that since $r' < 0$, we have $v_1^{(1)} > 0$ according to (3.8). The refined value is equal to $\lambda_1^{(1)} = 15.00847$, and the relative error is $|\lambda_1^{(1)} - \lambda_1| \lambda_1^{-1} = 5.7 \times 10^{-4}$, i.e. of the order of $\varepsilon^2$. Thus, despite the "roughness" of the choice of the test function $\psi_1$, satisfactory results of the estimation are obtained after the first iteration.

*Example* 4. We will briefly consider a similar problem for which the test function also differs considerably from the exact eigenfunction. We will take an Euler-type equation [6] with the function $r(x) = (1 + x)^{-2}, x \in [0, 1]$. We obtain the exact closed-form solution

$$\lambda_n = \frac{1}{4} + \left(\frac{\pi n}{\ln 2}\right)^2, \quad u_n(x) = \sqrt{1 + x} \sin\left(\frac{\pi n}{\ln 2} \ln(1 + x)\right)$$

$$\lambda_1 = \frac{1}{4} + \left(\frac{\pi}{\ln 2}\right)^2 = 20.79229, \quad u_1(x) = \sqrt{1+x}\sin\left(\frac{\pi}{\ln 2}\right)\ln(1+x)$$

For the usual elementary function $\psi_1 = \sin \pi x$ we obtain from formula (1.4) $\lambda_1^* = 22.22421$, which cannot be regarded as a good approximation. Carrying out the calculations as in Section 2, we obtain $\xi = 0.95459$, $\varepsilon = 4.5 \times 10^{-2}$; by Theorem 1 the lower limit is $\lambda_{1*} = \lambda_1^{(1)} = 20.25164$. From (3.8) and (3.9) we obtain the refined value $\lambda_1^{(1)} = 20.80330$. The absolute error is $\lambda_1^{(1)} - \lambda_1 \simeq 10^{-2}$, while the relative error is $(\lambda_{1*} - \lambda_1)\lambda_1^{-1} = 5 \times 10^{-4} \sim \varepsilon^2$, i.e. there is a considerable increase in the accuracy of the estimate.

We can similarly investigate examples which are more interesting from the applied point of view. Note that the proposed methods of estimating the eigenvalues and the approximate numerical–analytic solution of the Sturm–Liouville problems will be effective to an even greater extent if Eq. (1.1) is close to exactly integrable. In particular, if it is close to an equation with constant coefficients, one can construct systems of eigenvalues $\{\lambda_n(\varepsilon)\}$ and eigenfunctions $\{u_n(x, \varepsilon)\}$ with a specified degree of accuracy with respect to $\varepsilon$, and uniform with respect to the subscript $n$, $n = 1, 2, \ldots$ [17].

## 6. EXTENSION OF THE APPROACH TO OTHER CLASSES OF PROBLEM

We will discuss the possibility of developing the numerical–analytic method of Sections 2–4 for other types of boundary conditions. To fix our ideas we will consider rather briefly the case of boundary conditions of the second kind: $u'(0) = u'(1) = 0$. In order not to complicate the discussion, in Eq. (1.1) we will put $p(x) \equiv 1$, $q(x) \equiv 0$. This assumption corresponds to the mechanical model of a free string under tension with a linear density $r(x)$ that varies along the length, see Section 5. The approach described below can be extended to the more general case $q(x) \not\equiv 0$, $p = p(x)$. Thus, we will consider the Sturm–Liouville problem

$$u'' + \lambda r(x)\, u = 0, \quad u'(0) = u'(1) = 0 \tag{6.1}$$

It has the eigenvalue $\lambda_0 = 0$ and the corresponding eigenfunction $u_0 = 1$. It is required to determine (to estimate and refine) the next non-zero eigenvalue $\lambda_1$. We will compare the corresponding variational problem (see Section 1) with boundary-value problem (6.1). It is required to obtain the minimum of the functional with the additional isoperimetric conditions

$$J[u] = \int_0^1 u'^2 dx \to \min, \quad \Phi[u] = \|u\|^2 = 1, \quad \Phi_1[u] = (1, u)_r = 0 \tag{6.2}$$

The calculations are carried out using a scheme similar to that in Sections 1–3. Further, for clarity and to fix our ideas will take the function $r(x) = (1 + x^2)^{-2}$, $x \in [0, 1]$ of Example 3 considered above. For comparison we will write the exact closed-form solution

$$\lambda_n = \gamma_n^2 - 1, \quad \gamma = \operatorname{Arctg}\left(\operatorname{tg}\left(\frac{1}{4}\pi\gamma\right) - \gamma^{-1}\right), \quad \lambda_1 = 17.41687 \tag{6.3}$$

$$u_n(x) = \sqrt{1+x^2}\cos(\gamma_n \operatorname{arctg} x), \quad n = 1, 2, \ldots$$

As the test function $\psi_1$ we will take the simplest function $\psi_1 = \alpha + \cos \pi x$ from the class considered; the constant $\alpha$ is determined from the orthogonality condition (with weight $r(x)$) $\Phi_1[\psi_1] = 0$ (6.2). As a result we obtain the function $\psi_1$, which differs considerably from $u_1(x)$ (6.3), and the corresponding lower estimate of the first eigenvalue in accordance with Rayleigh's principle

$$\psi_1(x) = \cos \pi x - 0.27269, \quad \lambda_1 \leqslant \lambda_1^* = 18.06583 \tag{6.4}$$

We will solve the Cauchy problem by numerical integration of Eq. (6.1) with $\lambda = \lambda_1^*$ (6.4) and the conditions at the left end $v_1(0) = 1$, $v_1'(0) = 0$. Further, we will determine the quantity $\xi = 0.96875$ for which $v_1'(\xi) = 0$, and the small parameter $\varepsilon = 0.03125$. Using the approach described in Section 2 we obtain the lower limit $\lambda_{1*} = \lambda_1^* \xi^2 = 16.95440$. Using the perturbation method (see Section 3) we obtain the refined value $\lambda_1^{(1)} = 17.48420$, the relative error of which is $(\lambda_1^{-1} - \lambda_1)\lambda_1^{-1} = 4 \times 10^{-3} \sim \varepsilon^2$.

We will take the more complex test function $\psi_1$, which contains two harmonics

$$\psi_1(x) = c_1(\cos \pi x - 0.27269) + c_2(\cos 2\pi x + 0.00125)$$

It satisfies the condition $\Phi_1[\psi_1] = 0$ (6.2). The upper limit obtained by the Rayleigh–Ritz method is $\lambda_1^* = 17.41833$, which is considerably closer to the exact value of $\lambda_1$ given by (6.3) than that obtained in the single-coordinate approximation (6.4). Integration of the corresponding Cauchy problem (see above) leads to the values $\xi = 0.99993$, $\varepsilon = 7 \times 10^{-5}$. Calculation of the lower limit gives the required value $\lambda_{1*} = \lambda_1^* \xi^2 = 17.41575$, which is also rather close to the exact value. Calculating the mean, we obtain $\langle \lambda_1 \rangle - \lambda_1 = 0.00017$, which differs from the exact value by an amount $\langle \lambda_1 \rangle - \lambda_1) = 0.00017$, which can be assumed to be a highly accurate approximation. The relative error will be $(\langle \lambda_1 \rangle - \lambda_1)\lambda_1^{-1} = 10^{-5}$. Refinement of $\lambda_1$, i.e. calculation of $\lambda_1^{(1)}$, leads to an even smaller error $O(\varepsilon^2) \sim 10^{-8}$. Hence, as noted above, the accuracy of the calculation can be increased considerably and the amount of work involved in the calculations can be reduced by complicating the test function (see Example 2 in Section 5).

We can use a procedure similar to that described above to estimate and refine the eigenvalues and functions for $n \geqslant 2$, and also for boundary conditions of the third kind, mixed boundary conditions, etc.

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